RADC-TDR-62-67 15 January 1962

ELECTRON DIFFUSION IN A MULTIMODE CAVITY

By
A. D. MacDonald

TECHNICAL NOTE NO. 1

Contract No. AF 30(602)-2501

Prepared for

Rome Air Development Center
Air Force Systems Command
United States Air Force
Griffiss Air Force Base
New York

PALO ALTO LABORATORIES
MICROWAVE PHYSICS LABORATORY

GENERAL TELEPHONE AND ELECTRONICS LABORATORIES, INC.

GENERAL SYSTEM

RADC-TDR-62-67 15 January 1962

ELECTRON DIFFUSION IN A MULTIMODE CAVITY

by

A. D. MacDonald

GENERAL TELEPHONE AND ELECTRONICS LABORATORIES, INC.
MICROWAVE PHYSICS LABORATORY
1015 Corporation Way
Palo Alto, California

TECHNICAL NOTE NO. 1

Contract No. AF 30(602)-2501

Project No. 5561
Task No. 55237

Prepared For

Rome Air Development Center
Air Force Systems Command
United States Air Force

Griffiss Air Force Base
New York

ABSTRACT

An approximate solution of the diffusion equation describing electron diffusion in a multimode rectangular cavity is described. The results indicate that the electron concentration builds up in the spaces between the maxima of ionization. This means that the breakdown electric field is characteristic of a diffsuion length approximating that of the lowest diffusion mode.

LIST OF ILLUSTRATIONS

ELECTRON DIFFUSION IN A MULTIMODE CAVITY

There have been several studies of electron diffusion in cavities in which the non-uniform nature of the microwave electric field has complicated our understanding of the breakdown phenomena.[1,2] In a previous report to the Rome Air Development Center[3] an analysis of the problem was given using a variational treatment. The results were presented in the form of the ratio of an effective characteristic diffusion length, Λ_e, to the geometrical diffusion length, Λ, for different values of E/p, where E is the electric field at its maximum and p is the pressure. We have measured the breakdown field in a cavity of the type described in this report and find that experiment and theory do not agree. In this report we present a simpler theory of the breakdown phenomena and the results of the experiments, as well as the conclusions which we may draw from them.

We define the particle current density, Γ, by the equation

$$\Gamma = - \nabla (Dn), \tag{1}$$

where D is the diffusion coefficient and n is the electron concentration. The continuity equation is

$$\frac{\partial n}{\partial t} + \nabla \cdot \Gamma + P = 0, \tag{2}$$

where P is the production or loss rate at sources or sinks within the volume considered. We may consider breakdown to take place when the

production rate slightly exceeds the loss rate and $\partial n/\partial t$ may be set equal to zero.

Let $Dn = \psi$ and combine Equations (1) and (2). The result is

$$\nabla^2 \psi + P = 0. \tag{3}$$

The production rate within the cavity is determined by ionization and is, in fact, $n\nu_i$, where ν_i is the number of ionizations per second per electron. $n\nu_i$ is then equal to $\zeta E^2 \psi$, where ζ is the high frequency ionization coefficient, defined as ν_i/DE^2 (E is the rms value of the electric field).

Equation (3) then becomes

$$\nabla^2 \psi + \zeta E^2 \psi = 0. \tag{4}$$

The electron concentration vanishes on the walls of the cavity and this provides us with the necessary boundary conditions on Equation (4).

The cavity in which we are particularly interested is a rectangular parallelepiped resonant in the TM_{230} mode. The electric field is then given by the following expressions, where x, y, and z are the coordinates indicated in Figure 1.

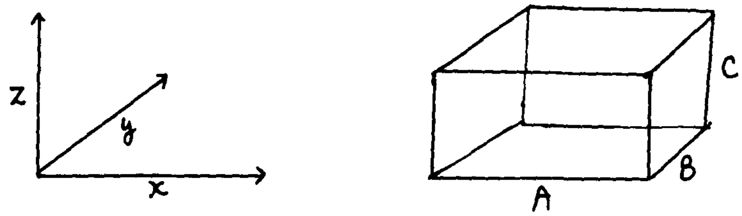

FIGURE 1

SCHEMATIC REPRESENTATION OF TM_{230} CAVITY

$$E_x = 0 \qquad\qquad E_y = 0$$
$$E_z = E_o \sin\frac{2\pi x}{A} \sin\frac{3\pi y}{B} \tag{5}$$

We use the approximation used by Herlin and Brown for this ionization coefficient, namely,

$$\zeta = \zeta_o \, (E/E_o)^{\beta - 2}, \tag{6}$$

where E_o is the field at the center of the cavity. Therefore, Equation (4) becomes

$$\nabla^2 \psi + \zeta_o E_o^2 \left(\sin\frac{2\pi x}{A} \sin\frac{3\pi y}{B}\right)^{\beta} \psi = 0. \tag{7}$$

Typical values of β range from about 8 to 16. A plot of $\sin^{\beta} x$ for two or three values of β is shown in Figure 2. This gives

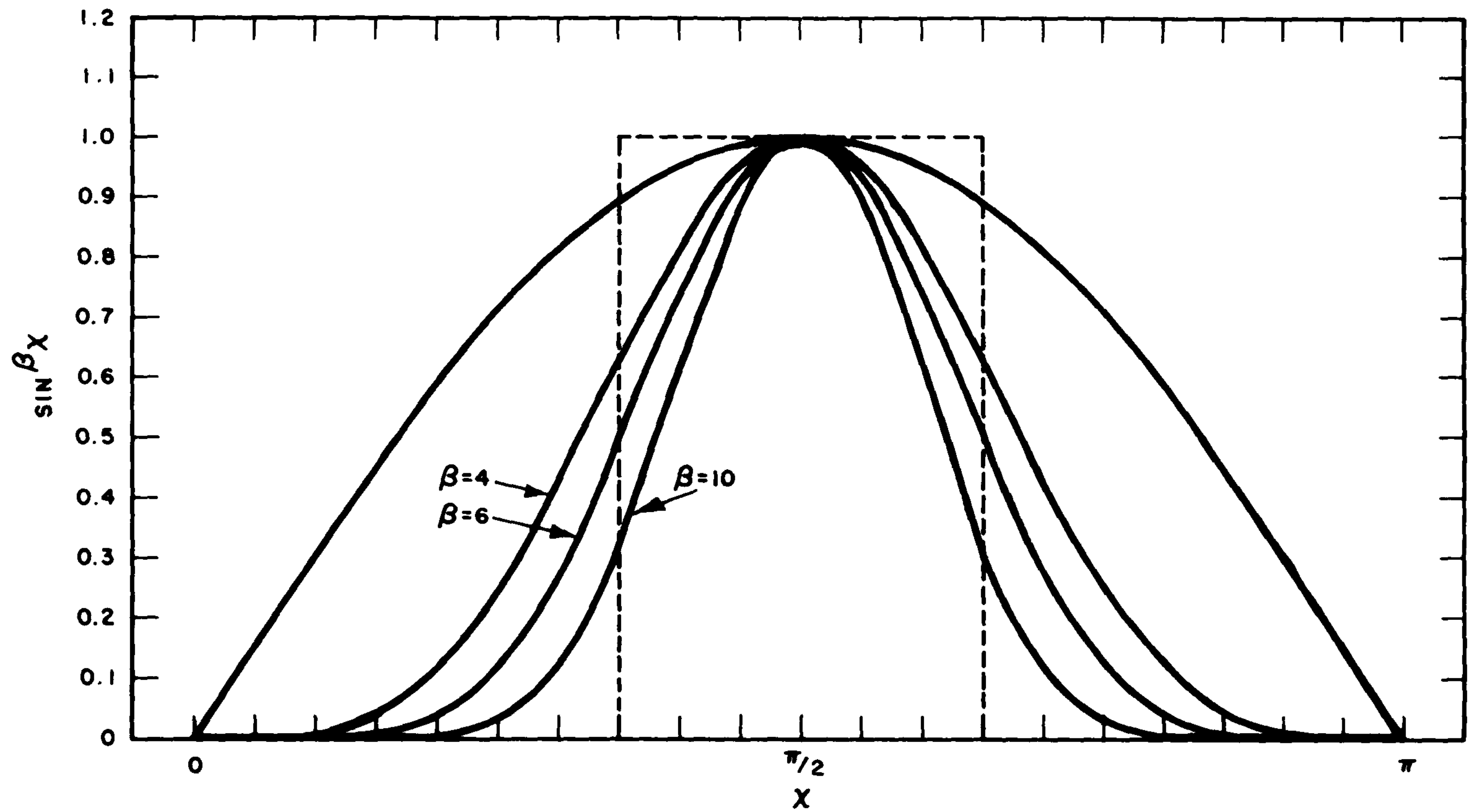

FIGURE 2

$\sin^{\beta} x$ AS A FUNCTION OF x

an indication of what a cross section of an ionization surface over the area of the cavity would look like. Inspection shows that the ionization is less than one-third of the maximum over a distance of two-thirds of the x axis. As a first approximation we shall use a step function to represent the ionization, and this is represented analytically by the following function:

$$\zeta E^2 = P, \text{ a constant} \qquad \begin{array}{r} .7a < x < 1.3a \\ 2.7a < x < 3.3a \\ .7b < y < 1.3b \\ 2.7b < y < 3.3b \\ 4.7b < y < 5.3b \end{array} \tag{8}$$

where A = 4a and B = 6b and is zero elsewhere.

The dotted lines in Figure 2 indicate how the approximation compares with the actual function.

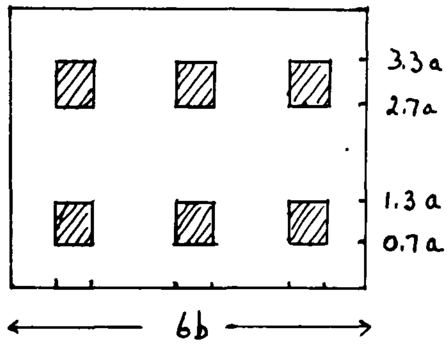

FIGURE 3

CROSS-SECTIONAL REPRESENTATION OF APPROXIMATION TO IONIZATION RATES

We can also represent this ionization function as viewed from the direction of the z axis in Figure 3. In this figure the cross-hatched areas represent those areas in which there is ionization.

Equation (7) then becomes

$$\nabla^2 \psi + P \psi = 0, \tag{9}$$

where $P = 0$ for most of the plane of Figure 3 and is a constant in the cross-hatched areas.

Because there is no variation of field in the z direction we may assume that the z variation of ψ is sinusoidal.

We let $\psi = g(x,y) \sin \frac{\pi z}{C}$, where g is a function of x and y but not of z.

$$\frac{1}{g} \left[\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right] - \frac{\pi^2}{C^2} + P = 0. \tag{10}$$

The solution of this equation with the type of ionization depicted in Figure 3 is extremely difficult because the equation may not be further separable and because of the nature of the boundary conditions.

However, we can find out a good deal about the nature of the physical processes going on by reducing the problem to one in which we assume that the ionization is a constant equal to P in the areas $.7a < x < 1.3a$ and $2.7a < x < 3.3a$, and zero elsewhere. This

is, of course, not very close to what happens; but insofar as we are dealing with variations in the electron concentration in the x direction, it is a good approximation, at least for some values of y. We can then find the manner of the variation in the x direction and infer that the y variation is similar functionally.

We get $g(x,y) = X(x)\,Y(y)$, and because there is no variation in the ionization in the Y direction we set $Y = \sin \frac{\pi y}{B}$. Equation (10) then becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{\pi^2}{B^2} - \frac{\pi^2}{C^2} + P = 0. \tag{11}$$

When P is greater than $\frac{\pi^2}{B^2} + \frac{\pi^2}{C^2}$, X is sinusoidal; and when P is zero, X is exponential. Therefore, the solutions are of the form

$$X = \cos\,(\mu x - \delta),$$

where there is ionization, and

$$X = \sinh \alpha x + k \cosh \alpha x \quad \text{elsewhere.}$$

These solutions lead to two conditions:

$$\alpha^2 = \frac{\pi^2}{B^2} + \frac{\pi^2}{C^2} \tag{12}$$

and

$$\mu^2 = P - \alpha^2. \tag{13}$$

We now write down the details of the x variation:

$$x < .7a \qquad X = d_1(\sinh \alpha x + k_1 \cosh \alpha x) \tag{14}$$

$$.7a < x < 1.3a \qquad X = \cos\left[\mu(x - a) - \delta\right] \tag{15}$$

$$1.3a < x < 2.7a \qquad X = d_2\left\{\cosh\left[\alpha(x - 2a)\right] + k_2 \sinh \alpha(x - 2a)\right\} \tag{16}$$

$$2.7a < x < 3.3a \qquad X = \cos\left[\mu(x - 3a) + \delta\right] \tag{17}$$

$$3.3a < x < 4.0a \qquad X = d_3\left[\sinh \alpha(x - 4a) + k_3 \cosh \alpha(x - 4a)\right] \tag{18}$$

This function must be continuous and have a continuous first derivative and it must vanish at the boundaries. Therefore, $k_1 = k_3 = 0$. Because of the symmetry the slope at $x = 2a$, the center of the x variation, is zero and, therefore, k_2 is zero. These boundary conditions lead to the following equations:

$$d_1 \sinh .7a\alpha = \cos(-.3a\mu - \delta) \tag{19}$$

$$d_1\alpha \cosh .7a\alpha = -\mu \sin(-.3a\mu - \delta) \tag{20}$$

$$d_2 \cosh(-.7a\alpha) = \cos(.3a\mu - \delta) \tag{21}$$

$$d_2\alpha \sinh(-.7a\alpha) = -\mu \sin(.3a\mu - \delta) \tag{22}$$

The dimensions of the cavity studied are $A = 4.91$ cm, $B = 6.36$ cm, and $C = 6.98$ cm. The value of the characteristic diffusion length for the lowest mode would then be $\Lambda = 1.08$ cm. From Equation (12) we find that $\alpha = .6683$/cm. Because we are dealing with a non-uniform electric field we are unable to specify P initially. P is

determined by the value of the high frequency ionization coefficient, ζ_o, and the value of the breakdown field, $\zeta_o E^2$, is equal to ν/D, which is equal to $1/\Lambda^2$ in the uniform field case and may be set equal to $1/\Lambda_e^2$, where Λ_e is some effective characteristic diffusion length in this case. As a first approximation to Λ_e we use the uniform field lowest mode Λ and then solve the problem by successive approximations, finding values of μ, d_1, d_2, and δ which are consistent with Equations (19) - (22). The values so obtained are as follows:

$$d_1 = 1.20$$

$$d_2 = .830$$

$$\mu = 1.37/\text{cm}$$

$$\delta = .253$$

The electron concentration is plotted as a function of x for these values in Figure 4.

On Figure 4 we have also plotted a sinusoidal curve with the same peak concentration.

We may now reverse the procedure and calculate an electron concentration profile along the y direction, and again calculate the various constants by successive approximations. The results which are not reproduced here are similar to those in the x direction, and the electron concentration profile has three peaks. The concentration gradient at the edge, which is the only number we shall use, is .782/cm. This value and the corresponding value for the x direction

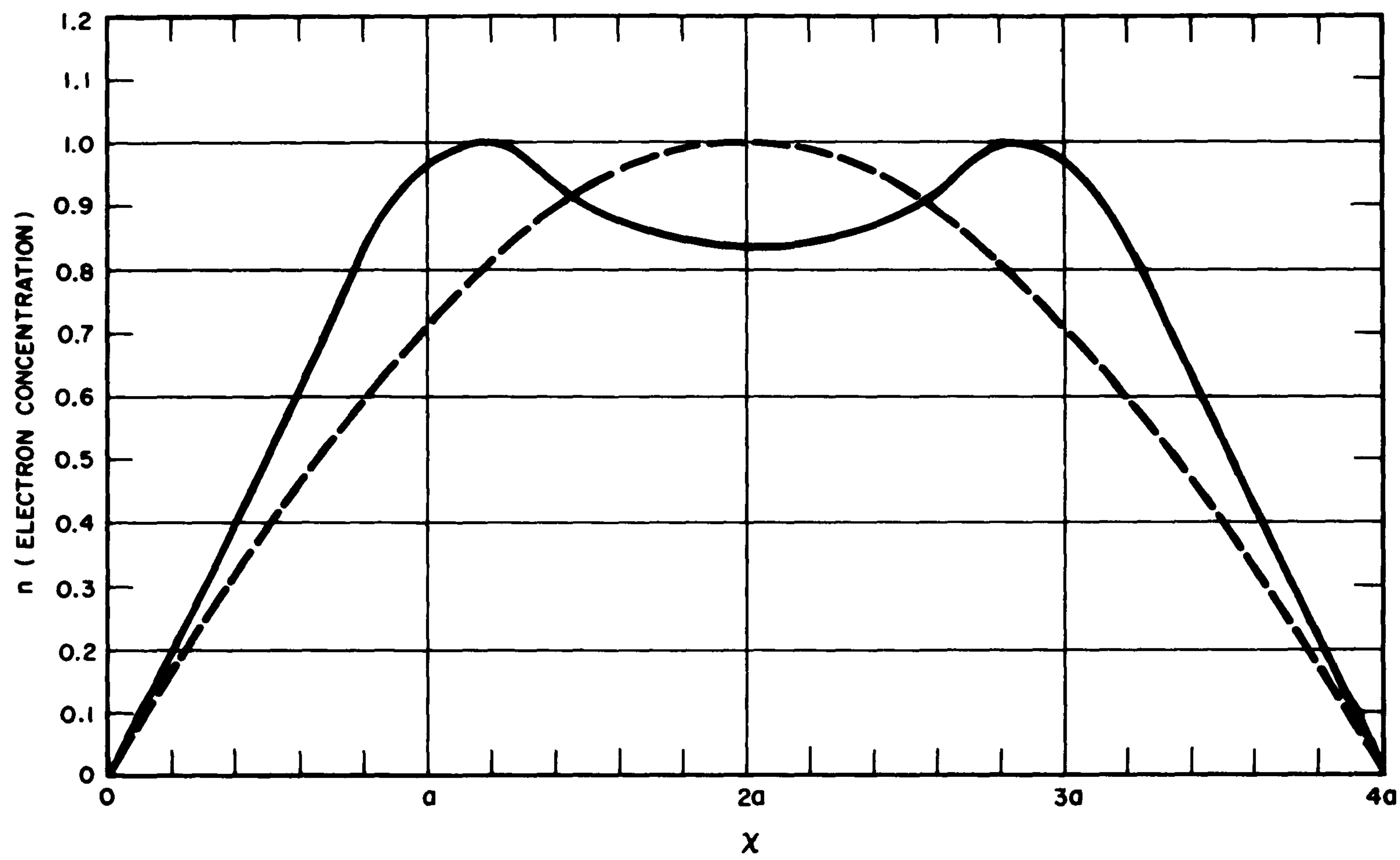

FIGURE 4

ELECTRON CONCENTRATION FROM EQUATIONS (14) THROUGH (18) COMPARED WITH sin x

are not sensitive to the actual size of the volume in which ionization occurs. Variations of factors of four in the size of the volume make very small changes in the result.

We now use these results to calculate an effective diffusion length. The rate of electron loss at any surface is determined by the concentration gradient, and one way of calculating the effective characteristic diffusion loss would be to calculate the total electron losses at all the surfaces and average the ionization rate over the whole cavity volume and then set the loss rate equal to the ionization rate. However, we may approximate this result closely by a much simpler procedure.

The ionization rate is proportional to the electric field raised to a very high power and, therefore, since the field need only change very little to make a large change in ionization rate, we will not need to average over the cavity volume. The characteristic diffusion length is ordinarily defined by the equation

$$1/\Lambda^2 = \gamma_1{}^2 + \gamma_2{}^2 + \gamma_3{}^2,$$

where γ_1, γ_2, and γ_3 are the eigen values of the function describing the electron concentration. In the lowest diffusion mode case they also represent the slope of the electron concentration at the central point of each surface; i.e., the concentration is $n = n_o \sin \gamma_1 x$ and $\frac{1}{n_o}\frac{\partial n}{\partial x} = \left[\gamma_1 \cos \gamma x\right]_{x=0} = \gamma_1$. We can, therefore, make a good approximation to the value of the effective diffusion length by determining the three γ's by finding $\frac{\partial n}{\partial x}$ at the center of each surface.

Inspection of Equation (14) gives us $\gamma_x = d_1\alpha = 1.00$, and the corresponding result for the y direction is $\gamma_y = \beta = .782$. Therefore, we find

$$\Lambda_e = .743 \text{ cm.}$$

Using this value of Λ_e, we calculate the high frequency ionization coefficient, ζ, and plot it as a function of E/p for various values of $p\lambda$, along with the values obtained for air in flat cylindrical cavities of dimensions such that the electric field is uniform. This is shown in Figure 5, and the original data from which ζ is calculated are shown in Figure 6. Also shown in Figure 6 are breakdown fields in the multimode cavity for nitrogen and hydrogen. The nature of the resulting curves shows that the results we obtained are reasonable, although at very high E/p the value of Λ_e appears definitely too low. High E/p corresponds to low pressures and at the low pressures it is likely that the electron concentration profile is closer to that of the lowest mode, and this would lead to a smaller value of ζ. The results obtained for the variational calculations, on the other hand, do not all agree with the results from the uniform field experiments.

The variational calculations were made with the assumption that the electron concentration was very low in the region between the ionization peaks, whereas the present calculation assumes, as we see from Figure 4, that the regions between high ionization points fill up with electrons because there is no place to which they can diffuse.

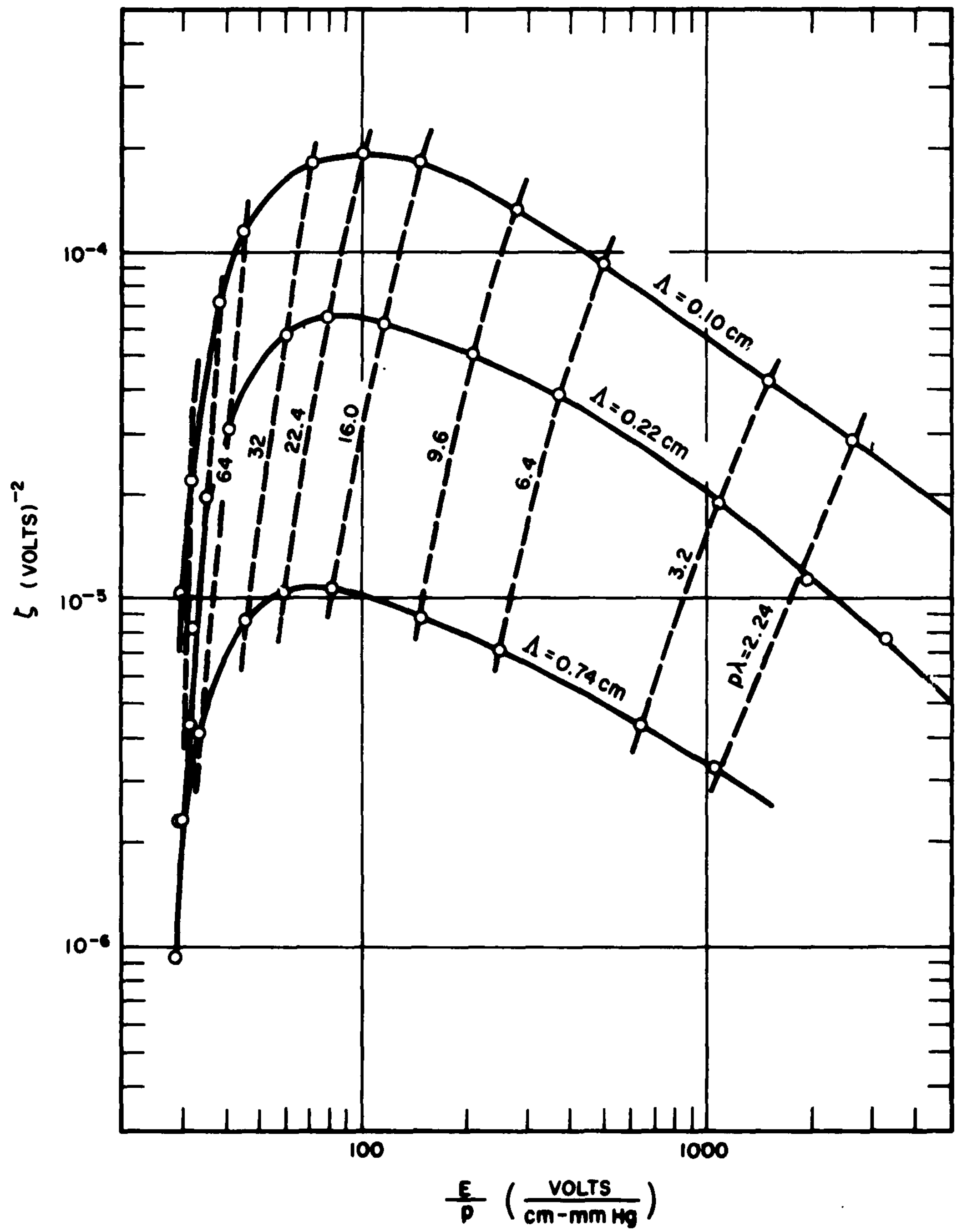

FIGURE 5

HIGH FREQUENCY IONIZATION COEFFICIENT ζ AS A FUNCTION OF E/p

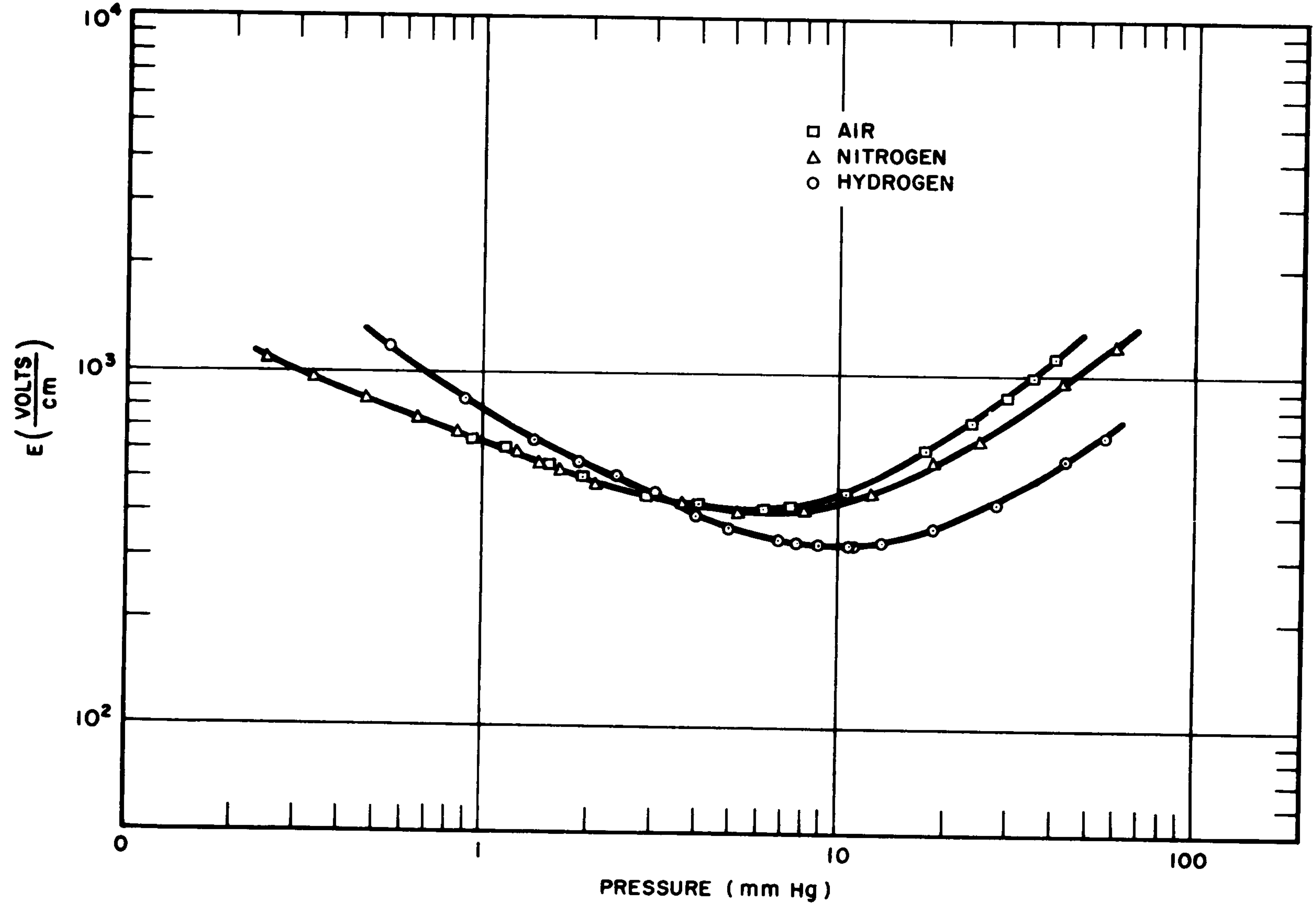

FIGURE 6

BREAKDOWN IN RECTANGULAR TM_{230} CAVITY AT 9.4 KMC

BIBLIOGRAPHY

1. M. A. Herlin and S. C. Brown, Physical Review, vol. 74, p. 1650 (1948).

2. A. D. MacDonald and S. C. Brown, Canadian Journal of Research, vol. A28, p. 168 (1950).

3. A. MacDonald, "Electrical Breakdown in the Atmosphere," Scientific Report No. 1, Contract AF30(602)-2261, Microwave Physics Laboratory, General Telephone and Electronics Laboratories,Inc., Palo Alto, California, 8 December 1960 (RADC-TN-61-16).